On the signature of coherent reliability systems: advances and applications

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Main points

✓ Basic concepts of structural Reliability

✓ Signature of reliability coherent structures

✓ Generating function approach for the signature vector

✓ Signature-based stochastic orderings between lifetimes of reliability structures

✓ Signature-based closure properties of aging classes under the formation of coherent systems

✓ Applications in the field of Reliability Engineering and Reliability Economics
\textbf{Structural Reliability: notations and definitions}

✓ Coherent system (CS) consisting of \( n \) components

Component's index

\[ x_i = \begin{cases} 
1, & \text{if } i \text{-component is working} \\
0, & \text{if } i \text{-component fails.} 
\end{cases} \]

Structure function of a CS \( \phi: \{0,1\}^n \rightarrow \{0,1\} \)

\[ \phi = \phi(x_1, x_2, \ldots, x_n) = \begin{cases} 
1, & \text{if the system is working} \\
0, & \text{if the system fails.} 
\end{cases} \]

- A reliability system is said to be coherent if
  - If \( x_i \leq y_i, i = 1, 2, \ldots, n \), then \( \phi(x_1, x_2, \ldots, x_n) \leq \phi(y_1, y_2, \ldots, y_n) \)
  - Each component is relevant

- A vector \( (x_1, x_2, \ldots, x_n) \) is said to be path vector if \( \phi(x_1, x_2, \ldots, x_n) = 1 \) and the corresponding set \( \{i : x_i = 1\} \subseteq \{1, 2, \ldots, n\} \) is called path set. (minimal path vector if additionally \( \phi(y) = 0 \), for \( y < x \))

- Reliability polynomial of a CS:
  \[ R = P(\phi(x_1, x_2, \ldots, x_n) = 1) \]
  \[ = 1 \cdot P(\phi(x_1, x_2, \ldots, x_n) = 1) + 0 \cdot P(\phi(x_1, x_2, \ldots, x_n) = 0) \]
  \[ = E(\phi(x_1, x_2, \ldots, x_n)) \]

Barlow & Proschan (1975)
Time to Failure ...

- **Component lifetimes**: $X_1, X_2, ..., X_n$

- **System's lifetime**: $T$

- **Ordered Component lifetimes**: $X_{1:n}, X_{2:n}, ..., X_{n:n}$

- **Reliability function**: $R(t) = P(T > t)$

- **Failure (Hazard) rate**: $- \frac{R'(t)}{R(t)} = \lim_{\Delta t \to 0} \frac{P(X_i \leq t + \Delta t \mid X_i > t)}{\Delta t}$

- **Mean Residual Lifetime**: $E(T - t \mid T > t) = \frac{1}{P(T > t)} \int_t^\infty P(T > x)dx$
The signature of a coherent system

✓ **Definition:** \( s_i(n) = P(T = X_{i,n}), \quad i = 1,2,...,n \) (Samaniego (1985))

✓ **Under i.i.d. scheme:**

- Signature depends only on the system structure and not on the (common) distribution.

- Signature’s \( i \)-th coordinate expresses the proportion of permutations \( \Delta_i \), among the \( n! \) likely permutations of \( X_1, X_2,..., X_n \) that result in system’s failure upon the occurrence of \( X_{i,n}, \quad i = 1,2,...,n \).

\[ s_i(n) = \frac{\Delta_i}{n!} \]

\[ s_i(n) = a_{n-i+1}(n) - a_{n-i}(n), \quad i = 1,2,...,n \]

\[ a_i(n) = r_i(n) / \binom{n}{i}, \text{ where } r_i(n) \text{ denotes the } \# \text{ of } \text{ path sets of size } i \]
An illustrative example

✓ Bridge structure

\[ \phi(x) = 1 - \prod_{i=1}^{M} (1 - \prod_{j \in P_j} x_j) \]

\[ P_j : \{1,2\}, \{4,5\}, \{1,3,5\}, \{4,3,2\} \]

\[ \phi(x) = 1 - (1 - x_1 x_2)(1 - x_4 x_5)(1 - x_1 x_3 x_5)(1 - x_4 x_3 x_2) \]
\[ = x_1 x_2 + x_2 x_3 x_4 - x_1 x_2 x_3 x_4 + x_1 x_3 x_5 - x_1 x_2 x_3 x_5 + x_4 x_3 \]
\[ - x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5 - x_2 x_3 x_4 x_5 + 2x_1 x_2 x_3 x_4 x_5 \]

\[ R = E(\phi(X)) = p_1 p_2 + p_2 p_3 p_4 - p_1 p_2 p_3 p_4 + p_1 p_3 p_5 - p_1 p_2 p_3 p_5 + p_4 p_5 - p_1 p_3 p_4 p_5 \]
\[ - p_1 p_3 p_4 p_5 - p_2 p_3 p_4 p_5 + 2p_1 p_2 p_3 p_4 p_5 \]

✓ Under i.i.d. scheme:

\[ p_i = p, \quad i = 1,2,...,5 \]

\[ R(p) = R_{BS} = 2p^2 + 2p^3 - 5p^4 + 2p^5 \]
An illustrative example (cont.)

✓ Bridge structure

The system fails at the 2\textsuperscript{nd} ordered component failure if one of the following failures occurs: \{1,4\} or \{2,5\}

\[ s_2 = P(T = X_{(2)}) = \frac{24}{5!} = \frac{1}{5} \]

The system fails at the 3\textsuperscript{rd} ordered component failure if one of the following failures occurs: \{1,3,5\} or \{4,3,2\} or \{1,X,2\}, \{2,X,1\}, \{X,1,2\}, \{X,2,1\}, \{4,Ψ,5\}, \{5,Ψ,4\}, \{Ψ,4,5\}, \{Ψ,5,4\}, where X=3,4,5 and Ψ=1,2,3.

\[ s_3 = P(T = X_{(3)}) = \frac{72}{120} = \frac{3}{5} \]

The system fails at the 4\textsuperscript{th} ordered component failure iff the last two components which remain at working state are \{1,2\} or \{4,5\}.

\[ s_4 = P(T = X_{(4)}) = \frac{4!}{5!} = \frac{1}{5} \]

\[ s_{BS} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0) \]
Computation of signature: a generating function approach

Proposition 1.

\[ \sum_{n=1}^{\infty} \sum_{i=1}^{n} i \binom{n}{i} s_i(n) t^i x^n = tx \frac{\partial H(x,t)}{\partial x} - t(t+1) \frac{\partial H(x,t)}{\partial t} \]

where \( H(x,t) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} r_{n-i}(n) t^i x^n \).

Proof sketch.

\[ a_i(n) = \binom{n}{i}^{-1} \quad r_i(n) \]

\[ i \binom{n}{i} s_i(n) = (n-i+1) r_{n-i+1}(n) - i r_{n-i}(n). \]

\[ s_i(n) = a_{n-i+1}(n) - a_{n-i}(n) \]
Computation of signature: a generating function approach

\[ \sum_{n=1}^{\infty} \sum_{i=1}^{n} i \binom{n}{i} s_i(n) t^i x^n = \sum_{n=1}^{\infty} \sum_{i=1}^{n} n r_{n-i+1}(n) t^i x^n - t^2 \sum_{n=1}^{\infty} \sum_{i=1}^{n} (i-1) r_{n-i+1}(n) t^{i-2} x^n \]

\[ t x \frac{\partial H(x,t)}{\partial x} - \sum_{n=1}^{\infty} n r_0(n) t^{n+1} x^n \]

\[ t^2 \frac{\partial H(x,t)}{\partial t} - \sum_{n=1}^{\infty} n r_0(n) t^{n+1} x^n \]

\[ t \frac{\partial H(x,t)}{\partial t} \]
Proposition 2. The double generating function of \( r_i(n) \) can be expressed by the aid of the generating function \( R(z;p) \) of system's reliability \( R_n(p) \) as

\[
\sum_{n=1}^{\infty} \sum_{i=1}^{n} r_i(n) t^i x^n = R(x(1+t); \frac{t}{1+t})
\]

Proof sketch. \( R_n(p) = \sum_{i=1}^{n} a_i(n) \binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} r_i(n) p^i q^{n-i} \)

\[ p = \frac{t}{1+t} \]

\[ z = x(1+t) \]

\[
R(z; p) = \sum_{n=1}^{\infty} R_n(p) z^n = \sum_{n=1}^{\infty} \sum_{i=1}^{n} r_i(n) \left( \frac{p}{q} \right)^i (qz)^n
\]
Computation of signature: a generating function approach

✓ Final result.

\[
\sum_{n=1}^{\infty} \sum_{i=1}^{n} \binom{n}{i} s_i(n) t^i x^n = tx \frac{\partial R(x(1+t); \frac{1}{1+t})}{\partial x} - t(t + 1) \frac{\partial R(x(1+t); \frac{1}{1+t})}{\partial t}
\]

✓ Having at hand the reliability of a structure or the corresponding generating function, we are able to compute its signature by the aid of the above formula.
Computation of signature: a generating function approach

General framework

✓ The reliability structure should be imbedded in a finite Markov chain

✓ The transition probability matrix $\Lambda$ should be determined and written in a suitable blocked form

✓ The generating function of system's reliability is given by

$$R(z; p) = \pi'_0 (I - z\Lambda)^{-1} u = \frac{1}{1 - z} - \frac{1}{\det(I - z\Lambda)} e_{i,N}$$

✓ Recent advances on the topic:

- Triantafyllou (2020). Mathematics
- Triantafyllou (2021). Int. Journal of Mathematical, Engineering and Management Sciences
Why is signature vector worth dealing with? (Part I)

✓ Signature is closely related to some well-known reliability concepts

✓ **Reliability function:**
  
  Kochar, Mukerjee & Samaniego (1999) (Naval Research Logistics)

  \[
  P(T > t) = \sum_{i=1}^{n} s_i(n) P(X_{i:n} > t)
  \]

  \[
  P(T > t) = \sum_{i=1}^{n} s_i(n) \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\overline{F(t)})^{n-j},
  \]

  where \( F \) denotes the distribution of \( X_i \)'s

✓ **Mean Residual Lifetime:**
  
  Eryilmaz, Koutras & Triantafyllou (2011) (Naval Research Logistics)

  \[
  MRL(t) = \frac{\sum_{i=1}^{n} s_i(n) P(X_{i:n} > t) MRL_{ss}(i,t)}{\sum_{i=1}^{n} s_i(n) P(X_{i:n} > t)},
  \]

  where \( MRL_{ss}(i,t) \) denotes the MRL function of a series system of size \( i \)
Why is signature vector worth dealing with? (Part II)

✓ Tool for establishing stochastic comparisons between lifetimes $X,Y$ of two reliability structures

✓ **Notations.** $F_X(x), F_Y(y) : \text{distribution functions of } X,Y$

$\quad r_X(t), r_Y(t) : \text{hazard rates of } X,Y$

✓ **Definition.** $X$ is said to be smaller than $Y$ in the hazard rate order ($X \leq_{hr} Y$) if $r_X(t) \geq r_Y(t)$ for all $t$.

✓ $r_X(t) \geq r_Y(t)$ if and only if the ratio $\frac{F_X(t)}{F_Y(t)}$ decreases for all $t$.

$P(Y-b > t \mid Y > b) \geq P(X-b > t \mid X > b)$, for all $t,b \geq 0$. 
Why is signature vector worth dealing with? (Part II)

Signature-based stochastic comparisons

**Definition.** Denote by $s_1 = (s_{11}, s_{12}, ..., s_{1n})$, $s_2 = (s_{21}, s_{22}, ..., s_{2n})$ the signature vectors of two reliability systems with lifetimes $X, Y$. If the ratio $\frac{\sum s_{2j}}{\sum s_{1j}}$ increases with respect to $i$, then $s_1 \leq_{hr} s_2$.

**Proposition 3.** If $s_1 \leq_{hr} s_2$ then $X \leq_{hr} Y$.

**Example.**

\[
\begin{align*}
s_1 &= \left(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0\right), \\
s_2 &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right)
\end{align*}
\]


Navarro & Rubio (2011) (Naval Research Logistics)
Proposition 4. The system lifetime $X$ is smaller than $Y$ in the hazard rate order $(X \leq_{hr} Y)$ if and only if the following condition holds true

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( s_{i}^{(1)}(n)s_{j}^{(2)}(n) - s_{i}^{(2)}(n)s_{j}^{(1)}(n) \right) \bar{F}_{i:n}(t) \bar{F}_{j:n}(t) \leq 0, \quad t \geq 0
$$

Proof sketch. $X \leq_{hr} Y$ iff

$$
A(t) = A(t) = \bar{F}_{X}'(t)\bar{F}_{Y}(t) - \bar{F}_{X}(t)\bar{F}_{Y}'(t) \leq 0
$$

Koutras , Triantafyllou & Eryilmaz (2016)
(Methodology and Computing in Applied Probability)
Additional signature-based sufficient and necessary conditions for establishing hazard rate and reverse hazard rate orderings are proved.

Signature-based stochastic orderings between well-known consecutive-type reliability structures are established.

**Proposition 5.** Let $X,Y$ be the lifetimes of a circular consecutive- $k$-out-of-$n$: $F$ and a circular $(n,m,k)$ system consisting both of the same $n$ components. It holds true that $(X \geq_{hr} Y)$ if

1. $X_{i:n} \geq_{hr} X_{j:n}$, for $i < j \leq m-1$
2. $s_i(n) / s_i(n-1)$ increases with respect to $i = 1, 2, ..., m-1$
3. $m \geq \lceil n / k \rceil$
Why is signature vector worth dealing with? (Part III)

✓ Tool for investigating the preservation of aging properties under the formation of a CS

Proposition 6. Assume that reliability system consists of \( n \) i.i.d. components. Then the system’s lifetime is IFR iff increases for \( x > 0 \).

\[
h(x) = \frac{\sum_{i=0}^{n-1} (n-i) \cdot s_{i+1}(n) \cdot \binom{n}{i} \cdot x^i}{\sum_{i=0}^{n-1} \left( \sum_{j=i+1}^{n} s_j(n) \right) \cdot \binom{n}{i} \cdot x^i}
\]

Drawback: The complicated form of \( h(x) \) makes the study of its monotonicity a difficult task.
Why is signature vector worth dealing with? (Part III)

✓ Tool for investigating the preservation of aging properties under the formation of a CS

Proposition 7. Let $n_0 (1 \leq n_0 < n)$ be the minimum number of working Components in a reliability structure such that the system can still operate successfully. If

$$s_{i_0}(n) > (n - i_0)s_{i_0 + 1}(n)$$

for $i_0 = n - n_0$, then the system does not preserve the IFR property.

Proof sketch.

$$h(x) = \frac{\sum_{i=0}^{n-1} (n-i) \cdot s_{i+1} \cdot \binom{n}{i} \cdot x^i}{\sum_{j=i+1}^{n} \left( \sum_{i=j}^{n} s_j \right) \cdot \binom{n}{i} \cdot x^i}$$

$$f(x) = h(1/x)$$

$$\lim_{x \to 0} f'(x) = \left( \alpha_{m-1} \beta_m - \alpha_m \beta_{m-1} \right) \beta_m^{-2}$$

Determine its sign

Triantafyllou & Koutras (2008)
(Probability in the Engineering and Information Science)
**Proof sketch of Proposition 7 (cont.)**

\[
\alpha_{m-1}\beta_m - \alpha_m\beta_{m-1} = (n-m+1)s_m(n) \binom{n}{m-1} \sum_{j=m+1}^{n} s_j(n) - (n-m)s_{m+1}(n) \binom{n}{m-1} \sum_{j=m}^{n} s_j(n)
\]

\[
s_{n-k_0+2}(n) = s_{n-k_0+3}(n) = \ldots = s_n(n) = 0
\]

\[
\alpha_{m-1}\beta_m - \alpha_m\beta_{m-1} = \binom{n}{i_0} \binom{n}{i_0-1} s_{i_0+1}(n) s_{i_0}(n) - (n-i_0) s_{i_0+1}(n)
\]

If \( s_{i_0}(n) > (n-i_0)s_{i_0+1}(n) \)

\[
\alpha_{m-1}\beta_m - \alpha_m\beta_{m-1} > 0
\]
Example. Let us consider the consecutive-2-out-of-n:F system.

For even $n$, we have $n_0 = \frac{n}{2}$, $i_0 = n - n_0 = \frac{n}{2}$

For odd $n$, we have $n_0 = \frac{n-1}{2}$, $i_0 = n - n_0 = \frac{n+1}{2}$

\[
\frac{s_{i_0}(n)}{s_{i_0+1}(n)} = \frac{s_{n_0}(n)}{s_{n_0+1}(n)} = \frac{n_0(n_0+1)\left(\frac{(n_0+1)(n_0+2)}{3!} - 1\right)}{n_0(n_0+1) - 1} > \frac{(n_0 + 1)(n_0 + 2)}{6} - 1 = \frac{(n_0 - 1)(n_0 + 4)}{6}.
\]

\[
\frac{s_i(n)}{s_{i+1}(n)} = \frac{s_{n_0+1}(n)}{s_{n_0+2}(n)} = \frac{n_0(n_0+1)\left(\frac{(n_0+1)(n_0+2)}{2} - 1\right)}{n_0(n_0+1) - 1} > \frac{(n_0 + 1)(n_0 + 2)}{2} - 1 = \frac{n_0(n_0 + 3)}{2} > n_0
\]

$n_0 \geq 5$

$s_{i_0}(n) > n_0s_{i_0+1}(n)$
A decision is made or an action is taken with a view toward balancing the performance of a system and its cost.

Three imposing challenges arise in seeking to address a problem in Reliability Economics analytically.

- Quantify the performance of the system
- Quantify the cost of the system
- Determine a criterion for comparing the systems of interest

We aim at justifying a particular formulation of the problem of finding the optimal system design relative to a specific family of criterion functions that take performance and cost into account.
Applications in Reliability Economics

- Focus on the problem of optimal system design

✓ **Main target:** Identify a system that strikes an appropriate balance between one’s positive expectations regarding its reliability, its cost and possible constraints.

✓ **Example.** Search for an “optimal” coherent system of order \( n \).

- For small \( n \), the entire collection of such systems is easy to enumerate.

- The number of distinct coherent systems of order \( n \) grows exponentially with \( n \).

- The problem of finding the best coherent system of a given order is, typically, a discrete optimization problem in which the space to be searched is huge.

- A second obstacle to the analytical treatment of this problem is the fact that there has been no obvious, manageable index with respect to which one might optimize.
A signature-based measure for optimizing both performance and cost

- **Performance measures**: \( R(t) \) or \( E(T) \)
  \[
  R(t) = \sum_{i=1}^{n} s_i(n)P(X_{i,n} > t) \quad \text{and} \quad E(T) = \sum_{i=1}^{n} s_i(n)E(X_{i,n})
  \]

- **Cost measure**: \( E(C) \)
  \[
  E(C) = \sum_{i=1}^{n} c_i s_i \quad \text{(e.g. salvage model)}
  \]

- **Criterion function**: \( m_r(s,a,c) = \frac{\sum_{i=1}^{n} a_i s_i}{r} \), \( r > 0 \)

Note that vectors \( a, c \) can be chosen arbitrarily within the context of two natural monotonicity constraints: \( 0 < a_1 < a_2 < \ldots < a_n, \ 0 < c_1 < c_2 < \ldots < c_n \)
Future potential

- Relax i.i.d assumption

- Applying the Markov chain imbedding approach, to develop recurrence relations for additional consecutive-type reliability structures.

- Investigate the aging preservation property under the formation of CS’s for different aging classes.
References


