# On the signature of coherent reliability systems: advances and applications


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**October 2020**

**Athens**

# Main points

✓ Basic concepts of structural Reliability

✓ Signature of reliability coherent structures

✓ Generating function approach for the signature vector

✓ Signature-based stochastic orderings between lifetimes of reliability structures

✓ Signature-based closure properties of aging classes under the formation of coherent systems

✓ Applications in the field of Reliability Engineering and Reliability Economics

# Structural Reliability: notations and definitions


## ✓ Coherent system (CS) consisting of *n* components

**Component's index**

$$x_i = \begin{cases} 1, & \text{if } \; i-\text{component is working} \\ 0, & \text{if } \; i-\text{component fails.} \end{cases}$$

**Structure function of a CS** $\left(\phi : \{0,1\}^n \to \{0,1\}\right)$

$$\phi = \phi(x_1, x_2, ..., x_n) = \begin{cases} 1, & \text{if } \; \text{the system is working} \\ 0, & \text{if } \; \text{the system fails.} \end{cases}$$

- A reliability system is said to be **coherent** if
  - If $x_i \le y_i, i = 1, 2, ..., n,$ then $\phi(x_1, x_2, ..., x_n) \le \phi(y_1, y_2, ..., y_n)$
  - Each component is relevant

$$\boxed{\phi(x) = 1 - \prod_{j=1}^{M} (1 - \prod_{i \in P_j} x_i)}$$

- A vector $(x_1, x_2, ..., x_n)$ is said to be path vector if $\phi(x_1, x_2, ..., x_n) = 1$ and the corresponding set $\{i : x_i = 1\} \subseteq \{1, 2, ..., n\}$ is called **path set**. (minimal path vector if additionally $\phi(\mathbf{y}) = 0$, for $\mathbf{y} < \mathbf{x}$)

- **Reliability polynomial** of a CS:

$$\begin{aligned} R &= P(\phi(x_1, x_2, ..., x_n) = 1) \\ &= 1 \cdot P(\phi(x_1, x_2, ..., x_n) = 1) + 0 \cdot P(\phi(x_1, x_2, ..., x_n) = 0) \\ &= E(\phi(x_1, x_2, ..., x_n)) \end{aligned}$$

Barlow & Proschan (1975)

# Time to Failure …


- ✓ Component lifetimes : $X_1, X_2, \ldots, X_n$
- ✓ System's lifetime : $T$
- ✓ Ordered Component lifetimes : $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$
- ✓ Reliability function: $R(t) = P(T > t)$
- ✓ Failure (Hazard) rate : $-\dfrac{R'(t)}{R(t)} = \lim\limits_{\Delta t \to 0} \dfrac{P(X_i \le t + \Delta t \mid X_i > t)}{\Delta t}$
- ✓ Mean Residual Lifetime: $E(T - t \mid T > t) = \dfrac{1}{P(T > t)} \int_t^{\infty} P(T > x)\,dx$

# The signature of a coherent system

✓ **Definition:** $s_i(n) = P(T = X_{i:n}), \quad i = 1,2,\ldots,n$ (Samaniego (1985))

✓ **Under *i.i.d.* scheme:**

- **Signature depends only on the system structure and not on the (common) distribution**
- **Signature's *i*-th coordinate expresses the proportion of permutations $\Delta_i$, among the *n*! likely permutations of $X_1, X_2, \ldots, X_n$ that result in system's failure upon the occurrence of $X_{i:n}, \; i = 1, 2, \ldots, n.$**

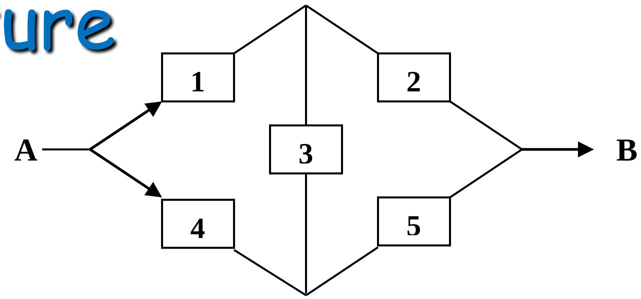
- $$s_i(n) = \frac{\Delta_i}{n!}$$

- $$s_i(n) = a_{n-i+1}(n) - a_{n-i}(n), \qquad i = 1,2,\ldots,n$$

$$\left( a_i(n) = r_i(n) \Big/ \binom{n}{i}, \text{ where } r_i(n) \textit{ denotes the } \# \text{ of path sets of size } i \right)$$

# An illustrative example

## ✓ Bridge structure




$$\phi(x) = 1 - \prod_{j=1}^{M} \left(1 - \prod_{i \in P_j} x_i\right) \xrightarrow{P_j \,:\, \{1,2\},\ \{4,5\},\ \{1,3,5\},\ \{4,3,2\}}$$

$$\begin{aligned}\phi(x) &= 1-(1-x_1x_2)(1-x_4x_5)(1-x_1x_3x_5)(1-x_4x_3x_2) \\ &= x_1x_2 + x_2x_3x_4 - x_1x_2x_3x_4 + x_1x_3x_5 - x_1x_2x_3x_5 + x_4x_5 \\ &\quad - x_1x_2x_4x_5 - x_1x_3x_4x_5 - x_2x_3x_4x_5 + 2x_1x_2x_3x_4x_5\end{aligned}$$

$$\begin{aligned}R = E(\phi(\mathbf{X})) &= p_1p_2 + p_2p_3p_4 - p_1p_2p_3p_4 + p_1p_3p_5 - p_1p_2p_3p_5 + p_4p_5 - p_1p_2p_4p_5 \\ &\quad - p_1p_3p_4p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5\end{aligned}$$

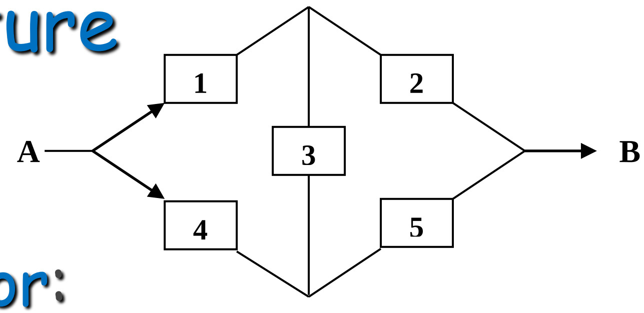
## ✓ Under *i.i.d.* scheme:

$$p_i = p, \quad i = 1,2,\ldots,5$$

$$R(p) = R_{BS} = 2p^2 + 2p^3 - 5p^4 + 2p^5$$

# An illustrative example (cont.)

## ✓Bridge structure




## ✓ Signature vector:

The system fails at the 2nd ordered component failure if one of the following failures occurs: {1,4} or {2,5}

$$2!\cdot 3! + 2!\cdot 3! = 24 \longrightarrow s_2 = P(T = X_{(2)}) = \frac{24}{5!} = \frac{1}{5}$$

The system fails at the 3rd ordered component failure if one of the following failures occurs: {1,3,5} or {4,3,2} or {1,X,2},{2,X,1}, {X,1,2},{X,2,1},{4,Ψ,5},{5,Ψ,4},{Ψ,4,5}, {Ψ,5,4}, where X=3,4,5 and Ψ=1,2,3.

$$\longrightarrow s_3 = P(T = X_{(3)}) = \frac{72}{120} = \frac{3}{5}$$

$$3!\cdot 2! + 3!\cdot 2! + 4\cdot 3\cdot 2! + 4\cdot 3\cdot 2! = 72$$

The system fails at the 4th ordered component failure iff the last two components which remain at working state are {1,2} or {4,5}.

$$\longrightarrow s_4 = P(T = X_{(4)}) = \frac{4!}{5!} = \frac{1}{5}$$

$$\boxed{\boldsymbol{s}_{BS}^{t} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)}$$

# Computation of signature: a generating function approach

Triantafyllou & Koutras (2008a)
*Probability in the Engineering and Informational Science*
Triantafyllou & Koutras (2008b)
*Advances in Mathematical Modeling Reliability*

✓ **Proposition 1.**

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} i\binom{n}{i} s_i(n)\, t^i x^n = tx\frac{\partial H(x,t)}{\partial x} - t(t+1)\frac{\partial H(x,t)}{\partial t}$$

**where** $H(x,t) = \sum_{n=1}^{\infty}\sum_{i=1}^{n} r_{n-i}(n)\, t^i x^n.$

**Proof sketch.**

$$\left.\begin{aligned} a_i(n) &= \binom{n}{i}^{-1} r_i(n) \\ s_i(n) &= a_{n-i+1}(n) - a_{n-i}(n) \end{aligned}\right\} \quad i\binom{n}{i} s_i(n) = (n-i+1)\, r_{n-i+1}(n) - i\, r_{n-i}(n).$$

# Computation of signature: a generating function approach

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} i\binom{n}{i} s_i(n)\, t^i x^n = \sum_{n=1}^{\infty}\sum_{i=1}^{n} n\, r_{n-i+1}(n) t^i x^n - t^2 \sum_{n=1}^{\infty}\sum_{i=1}^{n} (i-1) r_{n-i+1}(n) t^{i-2} x^n$$

$$-t \sum_{n=1}^{\infty}\sum_{i=1}^{n} i\, r_{n-i}(n) t^{i-1} x^n$$

$$tx \frac{\partial H(x,t)}{\partial x} - \sum_{n=1}^{\infty} n\, r_0(n)\, t^{n+1} x^n$$

$$t^2 \frac{\partial H(x,t)}{\partial t} - \sum_{n=1}^{\infty} n\, r_0(n)\, t^{n+1} x^n$$

$$t \frac{\partial H(x,t)}{\partial t}$$

# Computation of signature: a generating function approach

✓ **Proposition 2.** The double generating function of $r_i(n)$ can be expressed by the aid of the generating function *R*(*z;p*) of system's reliability $R_n(p)$ as

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} r_i(n)\, t^i x^n = R(x(1+t);\frac{t}{1+t})$$

**Proof sketch.**

$$R_n(p) = \sum_{i=1}^{n} a_i(n)\binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} r_i(n)\, p^i q^{n-i}$$

$$p = \frac{t}{1+t} \qquad z = x(1+t)$$

$$R(z;p) = \sum_{n=1}^{\infty} R_n(p) z^n = \sum_{n=1}^{\infty}\sum_{i=1}^{n} r_i(n)(\frac{p}{q})^i (qz)^n$$

# Computation of signature: a generating function approach

✓ Final result.

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} i\binom{n}{i} s_i(n)\, t^i x^n = tx\frac{\partial R(x(1+t);\frac{1}{1+t})}{\partial x} - t(t+1)\frac{\partial R(x(1+t);\frac{1}{1+t})}{\partial t}$$

✓ Having at hand the reliability of a structure or the corresponding generating function, we are able to compute its signature by the aid of the above formula.

# Computation of signature: a generating function approach

## General framework

✓ The reliability structure should be imbedded in a finite Markov chain

✓ The transition probability matrix Λ should be determined and written in a suitable blocked form

✓ The generating function of system's reliability is given by

$$R(z;p) = \boldsymbol{\pi}_0'\left(I - z\Lambda\right)^{-1}\mathbf{u} = \frac{1}{1-z} - \frac{1}{\det(I - z\Lambda)} e_{1,N}$$

✓ Recent advances on the topic:

- Triantafyllou (2020). *Mathematics*
- Triantafyllou (2021). *Int. Journal of Mathematical, Engineering and Management Sciences*

# Why is signature vector worth dealing with? (Part I)

✓ Signature is closely related to some well-known reliability concepts

✓ **Reliability function:**

Kochar, Mukerjee & Samaniego (1999)
*(Naval Research Logistics)*

$$P(T>t)=\sum_{i=1}^{n} s_i(n)P(X_{i:n}>t)$$

$$P(T>t)=\sum_{i=1}^{n} s_i(n)\sum_{j=0}^{i-1}\binom{n}{j}(F(t))^{j}(\overline{F}(t))^{n-j},$$

where $F$ denotes the distribution of $X_i$'s

✓ **Mean Residual Lifetime:**

Eryilmaz, Koutras & Triantafyllou (2011)
*(Naval Research Logistics)*

$$MRL(t)=\frac{\sum_{i=1}^{n} s_i(n)P(X_{i:n}>t)MRL_{SS}(i,t)}{\sum_{i=1}^{n} s_i(n)P(X_{i:n}>t)},$$

where $MRL_{SS}(i,t)$ denotes the MRL function of a series system of size $i$

# Why is signature vector worth dealing with? (Part II)


✓Tool for establishing stochastic comparisons between lifetimes X,Y of two reliability structures

✓ **Notations.** $F_X(x), F_Y(y)$ : distribution functions of X,Y

$r_X(t), r_Y(t)$ : hazard rates of X,Y

✓ **Definition.** **X is said to be smaller than Y in the hazard rate order** $(X \leq_{hr} Y)$ **if** $r_X(t) \geq r_Y(t)$ **for all t.**

✓ $r_X(t) \geq r_Y(t)$ **if and only if the ratio** $\bar{F}_X(t)/\bar{F}_Y(t)$ **decreases for all t.**

$$P(Y - b > t \mid Y > b) \geq P(X - b > t \mid X > b), \text{ for all } t, b \geq 0.$$

# Why is signature vector worth dealing with? (Part II)


## Signature-based stochastic comparisons

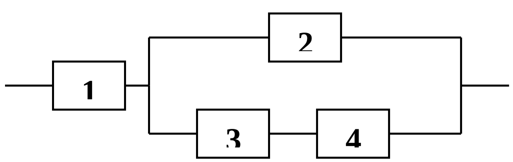
✓ **Definition.** Denote by $\mathbf{s}_1 = (s_{11}, s_{12}, \ldots, s_{1n})$, $\mathbf{s}_2 = (s_{21}, s_{22}, \ldots, s_{2n})$ the signature vectors of two reliability systems with lifetimes X,Y.
If the ratio $\sum_{j=i}^{n} s_{2j} / \sum_{j=i}^{n} s_{1j}$ increases with respect to *i*, then $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$.

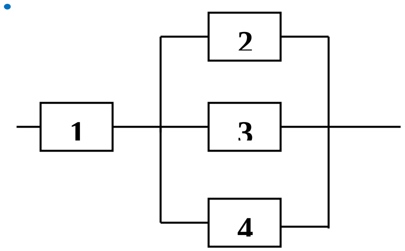
✓ **Proposition 3.** If $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$ then $X \leq_{hr} Y$.

Navarro, Rubio & Sandoval (2005)
(*Statistics & Probability Letters*)

Navarro & Rubio (2011)
(*Naval Research Logistics*)

Example.




$$\mathbf{s}_1 = \left(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0\right),$$

$$\mathbf{s}_2 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right)$$

$$\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$$

# Why is signature vector worth dealing with? (Part II)


## Signature-based stochastic comparisons between lifetimes X,Y of two reliability structures (cont.)

**Proposition 4.** **The system lifetime *X* is smaller than *Y* in the hazard rate order** $(X \le_{hr} Y)$ **if and only if the following condition holds true** $(\bar{F}_r(t) = P(T_r > t),\ r = X, Y)$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(s_i^{(1)}(n)s_j^{(2)}(n) - s_i^{(2)}(n)s_j^{(1)}(n)\right)\bar{F}'_{i:n}(t)\bar{F}_{j:n}(t) \le 0,\ \ t \ge 0$$

**Proof sketch.** $X \le_{hr} Y$ **iff** $A(t) = \bar{F}'_X(t)\bar{F}_Y(t) - \bar{F}_X(t)\bar{F}'_Y(t) \le 0$

$$\left.\begin{aligned}\bar{F}_X(t) &= \sum_{i=1}^{n} s_i^{(1)}(n)\bar{F}_{i:n}(t)\\ \bar{F}'_X(t) &= \sum_{i=1}^{n} s_i^{(1)}(n)\bar{F}'_{i:n}(t)\end{aligned}\right\} \quad A(t) = \left(\sum_{i=1}^{n} s_i^{(1)}(n)\bar{F}'_{i:n}(t)\right)\left(\sum_{j=1}^{n} s_j^{(2)}(n)\bar{F}_{j:n}(t)\right) - \left(\sum_{j=1}^{n} s_j^{(1)}(n)\bar{F}_{j:n}(t)\right)\left(\sum_{i=1}^{n} s_i^{(2)}(n)\bar{F}'_{i:n}(t)\right)$$

**Koutras , Triantafyllou & Eryilmaz (2016)**
*(Methodology and Computing in Applied Probability)*

# Why is signature vector worth dealing with? (Part II)

IMPORTANT

Koutras , Triantafyllou & Eryilmaz (2016)
*(Methodology and Computing in Applied Probability)*

✓ Additional signature-based sufficient and necessary conditions for establishing hazard rate and reverse hazard rate orderings are proved.

✓Signature-based stochastic orderings between well-known consecutive-type reliability structures are established.

**Proposition 5.** Let X,Y be the lifetimes of a circular consecutive- *k*-out-of-*n*: *F* and a circular (*n*,*m*,*k*) system consisting both of the same *n* components. It holds true that $(X \geq_{hr} Y)$ if

$X_{i:n} \geq_{hr} X_{j:n},\ \ for\ i < j \leq m-1$

$s_i(n)/s_i(n-1)$ increases with respect to $i = 1, 2, \ldots, m-1$

$m \geq [n/k]$

# Why is signature vector worth dealing with? (Part III)


✓Tool for investigating the preservation of aging properties under the formation of a CS

Proposition 6. Assume that reliability system consists of $n$ IFR i.i.d. components. Then the system's lifetime is IFR iff

$$h(x)=\frac{\sum_{i=0}^{n-1}(n-i)\cdot s_{i+1}(n)\cdot\binom{n}{i}\cdot x^{i}}{\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n}s_{j}(n)\right)\cdot\binom{n}{i}\cdot x^{i}}$$

increases for x>0.

Drawback: The complicated form of h(x) makes the study of its monotonicity a difficult task.

# Why is signature vector worth dealing with? (Part III)


## ✓Tool for investigating the preservation of aging properties under the formation of a CS

**Proposition 7. Let $n_0\ (1 \le n_0 < n)$ be the minimum number of working Components in a reliability structure such that the system can still operate successfully. If**

Triantafyllou & Koutras (2008)
*(Probability in the Engineering and Information Science)*

$$s_{i_0}(n) > (n - i_0) s_{i_0+1}(n)$$

**for $i_0 = n - n_0$, then the system does not preserve the IFR property.**

**Proof sketch.**

$$h(x) = \frac{\sum_{i=0}^{n-1} \underbrace{(n-i)\cdot s_{i+1}\cdot \binom{n}{i}}_{\alpha_i} \cdot x^i}{\sum_{i=0}^{n-1} \underbrace{\left(\sum_{j=i+1}^{n} s_j\right)\cdot \binom{n}{i}}_{\beta_i} x^i} = \frac{\sum_{i=0}^{m} \alpha_i x^i}{\sum_{i=0}^{m} \beta_i x^i},\ x > 0 \xrightarrow{f(x) = h(1/x)} \lim_{x \to 0} f'(x) = (\alpha_{m-1}\beta_m - \alpha_m \beta_{m-1}) \beta_m^{-2}$$

**Determine its sign**

# Why is signature vector worth dealing with? (Part III)

IMPORTANT

## Proof sketch of Proposition 7 (cont. )

**Triantafyllou & Koutras (2008)**
*(Probability in the Engineering and Information Science)*

$$\alpha_{m-1}\beta_m - \alpha_m\beta_{m-1} = (n-m+1)s_m(n)\binom{n}{m-1}\binom{n}{m}\sum_{j=m+1}^{n} s_j(n) - (n-m)s_{m+1}(n)\binom{n}{m}\binom{n}{m-1}\sum_{j=m}^{n} s_j(n)$$

$$s_{n-n_0+2}(n) = s_{n-n_0+3}(n) = \ldots = s_n(n) = 0$$

$$\alpha_{m-1}\beta_m - \alpha_m\beta_{m-1} = \binom{n}{i_0}\binom{n}{i_0-1} s_{i_0+1}(n)\left(s_{i_0}(n) - (n-i_0)s_{i_0+1}(n)\right)$$

If $s_{i_0}(n) > (n-i_0)s_{i_0+1}(n)$

$$\alpha_{m-1}\beta_m - \alpha_m\beta_{m-1} > 0$$

# Why is signature vector worth dealing with? (Part III)

Triantafyllou & Koutras (2008)
*(Probability in the Engineering and Information Science)*

**Example.** **Let us consider the consecutive-2-out-of-n:F system.**

For even $n$, we have $n_0 = \dfrac{n}{2}, \ i_0 = n - n_0 = \dfrac{n}{2}$

$$\frac{s_{i_0}(n)}{s_{i_0+1}(n)} = \frac{s_{n_0}(n)}{s_{n_0+1}(n)} = \frac{n_0(n_0+1)\left(\dfrac{(n_0+1)(n_0+2)}{3!}-1\right)}{n_0(n_0+1)-1} > \frac{(n_0+1)(n_0+2)}{6} - 1 = \frac{(n_0-1)(n_0+4)}{6}.$$

$$\xrightarrow{n_0 \geq 5}$$

$$s_{i_0}(n) > n_0 s_{i_0+1}(n)$$

For odd $n$, we have $n_0 = \dfrac{n-1}{2}, \ i_0 = n - n_0 = \dfrac{n+1}{2}$

$$\frac{s_{i_0}(n)}{s_{i_0+1}(n)} = \frac{s_{n_0+1}(n)}{s_{n_0+2}(n)} = \frac{n_0(n_0+1)\left(\dfrac{(n_0+1)(n_0+2)}{2}-1\right)}{n_0(n_0+1)-1} > \frac{(n_0+1)(n_0+2)}{2} - 1 = \frac{n_0(n_0+3)}{2} > n_0$$

$$\xrightarrow{n_0 \geq 1}$$

$$s_{i_0}(n) > n_0 s_{i_0+1}(n)$$

# Applications in Reliability Economics


✓A decision is made or an action is taken with a view toward balancing the performance of a system and its cost.

✓ Three imposing challenges arise in seeking to address a problem in Reliability Economics analytically.

- ❑ Quantify the performance of the system
- ❑ Quantify the cost of the system
- ❑ Determine a criterion for comparing the systems of interest

✓ We aim at justifying a particular formulation of the problem of finding the optimal system design relative to a specific family of criterion functions that take performance and cost into account.

# Applications in Reliability Economics

❑ **Focus on the problem of optimal system design**

✓**Main target:** Identify a system that strikes an appropriate balance between one's positive expectations regarding its reliability, its cost and possible constraints.

✓**Example**. Search for an "optimal" coherent system of order *n*.

- For small n, the entire collection of such systems is easy to enumerate.

- The number of distinct coherent systems of order *n* grows exponentially with *n*.

- The problem of finding the best coherent system of a given order is, typically, a discrete optimization problem in which the space to be searched is huge.
- A second obstacle to the analytical treatment of this problem is the fact that there has been no obvious, manageable index with respect to which one might optimize.

# Applications in Reliability Economics

✓ **A signature-based measure for optimizing both performance and cost**

- Performance measures: $R(t)$ or $E(T)$

$$R(t) = \sum_{i=1}^{n} s_i(n) P(X_{i:n} > t) \qquad E(T) = \sum_{i=1}^{n} s_i(n) E(X_{i:n})$$

- Cost measure: $E(C)$ $\quad E(C) = \sum_{i=1}^{n} c_i s_i$ (e.g. salvage model)

- Criterion function:

$$m_r(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \frac{\sum_{i=1}^{n} a_i s_i}{\left(\sum_{i=1}^{n} c_i s_i\right)^r}, \ r > 0$$

Note that vectors **a**,**c** can be chosen arbitrarily within the context of two natural monotonicity constraints: $0 < a_1 < a_2 < \ldots < a_n,\ 0 < c_1 < c_2 < \ldots < c_n$

# Future potential

- Relax *i.i.d* assumption

- Applying the Markov chain imbedding approach, to develop recurrence relations for additional consecutive-type reliability structures.

- Investigate the aging preservation property under the formation of CS's for different aging classes.

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